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Adiabatic theory for the population distribution in the evolutionary minority game

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We study the evolutionary minority game (EMG) using a statistical mechanics approach. We derive a theory for the steady-state population distribution of the agents. The theory is based on an ''adiabatic approximation'' in which short time fluctuations in the population distribution are integrated out to obtain an effective equation governing the steady-state distribution. We discover the mechanism for the transition from segregation (into opposing groups) to clustering (towards cautious behaviors). The transition is determined by two generic factors: the market impact (of the agents' own actions) and the short time market inefficiency (arbitrage opportunities) due to fluctuations in the numbers of agents using opposite strategies. A large market impact favors ''extreme'' players who choose fixed opposite strategies, while large market inefficiency favors cautious players. The transition depends on the number of agents (N) and the effective rate of strategy switching. When N is small, the market impact is relatively large; this favors the extreme behaviors. Frequent strategy switching, on the other hand, leads to a clustering of the cautious agents.

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Complex adaptive systems are ubiquitous in social, biological and economic sciences. In these systems agents adapt to the changes in the global environment, which are induced by the actions of the agents themselves. Despite the apparent complexity of these systems some generic features in their collective behaviors can be studied using models and techniques of statistical physics. Of particular interest is the so-called minority game proposed by Challet and Zhang [1,2], which models systems in which the agents have no direct interaction but compete to be in the minority. The game evolves as the agents modify their behaviors (strategies) based on the past experiences. Examples of such systems include financial markets [3], rush-hour traffic [4], and ecological systems. In the context of demand and supply in economic systems, the idea of the minority game is particularly relevant. If the demand is larger than the supply, the price of the goods will increase; this benefits the sellers who are in the minority. Many agent based models of economic systems and financial markets indeed incorporate the essence of the minority game.

The key question in the study of the agent-based models is how evolution changes the behaviors of the agents. In the context of a simple evolutionary minority game (EMG), Johnson *et al.* found that the agents universally self-segregate into two opposing extreme groups [5]. Hod and Nakar, on the other hand, claimed that a clustering of cautious agents emerges in a ''tough environment'' where the penalty for losing is greater than the reward for winning [8,9]. To understand the mechanism for the transition from segregation to clustering, we give a detailed statistical mechanical analysis of the EMG limited to three groups of agents: two opposing groups and one cautious group. We find that the population distribution of the agents can be studied using an adiabatic approximation, in which the short-term fluctuations of the market inefficiencies (arbitrage opportunities) are integrated out to obtain an equation for the steady state distribution of the agents.

We first briefly describe the EMG model. There are N (odd number) agents. At each round they choose to enter room 0 (sell a stock or choose route A) or room 1 (buy a stock or choose route B). At the end of each round the agents in the room with fewer agents (in the minority) win a point; while the agents in the room with more agents (in the majority) lose a point. The winning room numbers (0 or 1) are recorded, and they form a historical record of the game. All agents share the common memory containing the outcomes from the most recent occurrences of all 2^m possible bit strings of length m. The basic strategy is derived from the common memory. Given the current m-bit string, the basic strategy is simply to choose the winning room number after the most recent pattern of the same m-bit string in the historical record. To use the basic strategy is thus to follow the trend. In the EMG each agent is assigned a probability p: he will adopt the basic strategy with probability p and adopt the opposite of the basic strategy with probability $1-p$. The agents with $p=0$ or 1 are ''extreme'' players, while the agents with $p=1/2$ are cautious players. The game and its outcomes evolve as less successful agents modify their p values. This is achieved by allowing the agents with the accumulated wealth less than d ($d<0$) to change their p values. In the original EMG model, the new p value is chosen randomly in the interval of width Δp centered around its original p value. His wealth is reset to zero and the game continues.

Johnson and co-workers showed that the agents self-segregate into two opposing extreme groups with $p\sim 0$ and $p\sim 1$ [5–7]. This conclusion is very robust; it does not depend on N, d, Δp, m, or the initial distribution of p. The final distribution always has symmetric U-shape. Thus, in order to succeed in a competitive society the agent must take extreme positions (either always follows the basic strategy or goes against it). This behavior can be explained by the market impact of the agents' own actions which largely penalizes the cautious agents [7]. However, Hod and Nakar later found that the above conclusion is only robust when the reward-to-

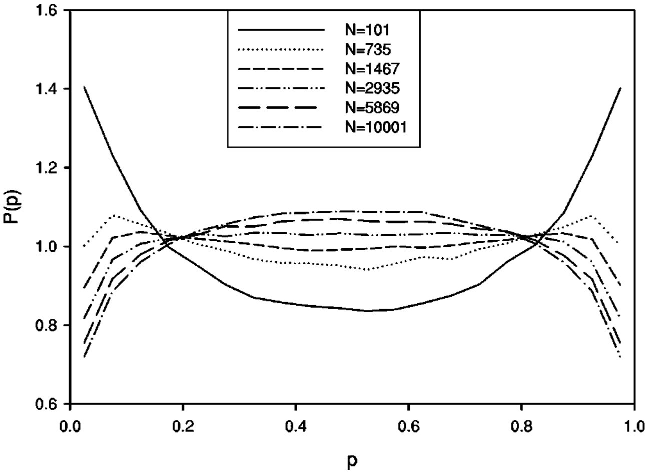

FIG. 1. The distribution $P(p)$ for $R=0.971$ and $d=-4$. A set of values of $N=101$, 735, 1467, 2935, 5869, and 10001 are used. The distribution is obtained by averaging over 100 000 time steps and ten independent runs.

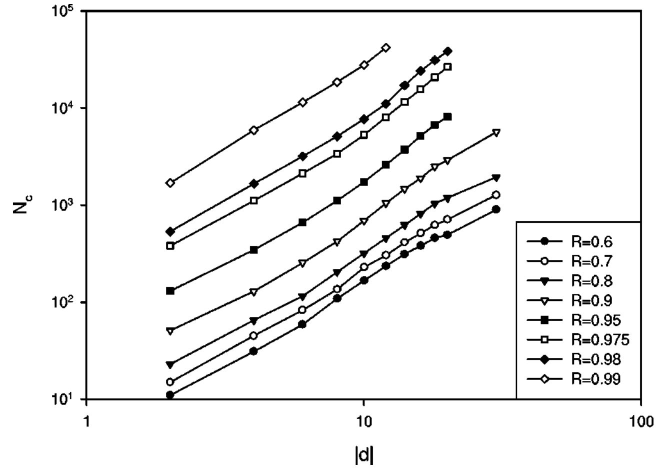

FIG. 2. The critical value N vs $|d_c|$ for $R=0.6$, 0.7, 0.8, 0.9, 0.95, 0.975, 0.98, and 0.99.

fine ratio $R\geqslant 1$. When $R<1$ there is tendency for the agents to cluster towards cautious behaviors and the distribution of the p value, $P(p)$, may evolve to an inverted-U shape with the peak at the middle (in some intermediate cases M-shaped distributions are also observed). To explain the clustering of cautious agents, Hod gave a phenomenological theory relating the accumulated wealth reduction to a random walk with time-dependent oscillating probabilities [10]. However, the dynamical mechanism for the phase transition is still lacking.

We have performed extensive simulations of the EMG for a wide range of the values of the parameters, N,R, and d. Our numerical results show that the transition from segregation to clustering is generic for $R<1$. The transition depends on all three parameters, N, R, and d. Figure 1 shows the distribution $P(p)$ for $R=0.971$, $d=-4$, and $N=101$, 735, 1467, 2935, 5869, and 10001. For a given $R(<1)$ and d, we observe a transition from segregation to clustering as the number of agents N increases. The shape of the distribution $P(p)$ changes from a U shape to an inverted-U shape [near the transition point $P(p)$ has an M shape]. The standard deviation σ_p of the distribution decreases as N increases. We define the critical value N_c as the value when σ_p is equal to the standard deviation of the uniform distribution, i.e. when $\sigma_p^2=\int_0^1(p-1/2)^2P(p)dp$ equal to 1/12. Our results can be summarized by the general expression for the critical value

$$N_c=\left[\frac{|d|}{A(1-R)}\right]^2,$$

where A is a constant of the order one. Alternatively one might view the transition by varying d with fixed N and R. As $|d|$ increases the system changes from clustering to segregation. The critical value is then given by $|d_c|=A(1-R)\sqrt{N}$. Figure 2 plots N_c vs $|d|$ for various values of R. When $R\to 1$ the clustering only occurs for either very large N or very small $|d|$. At $R=1$ the clustering disappears and the segregation to extreme behaviors becomes robust.

Hod and Nakar explained that $R<1$ corresponds to difficult situations or tough environments in which the agents tend to be confused and indecisive and thus become cautious. We find that the effective rate of strategy switching (which depends on both R and d) affects the distribution of the agents more directly. For $R<1$ the agent switches its strategy every $2|d|/(1-R)$ time steps on average. When R or $|d|$ is small, the agents have less patience and switch their strategies more frequently; this, as we shall explain below, causes large market inefficiency and thus favors cautious agents. It is the rapid evolution that makes the agents "confused" and "indecisive." On the other hand, when the number of agents is small, the market impact becomes large. Take, for example, a population consisting of only three agents with $p=0$, 1/2, and 1, respectively. The cautious agent (with $p=1/2$) always loses because he is always in the majority, while the extreme agents are in the majority half of the times. In this case the cautious agent experiences the full market impact of his own action. Indeed our data show that when N is small enough the segregation to extreme behaviors dominates.

We now show that the mechanism for clustering around $p=1/2$ and the transition from clustering to segregation can be understood clearly from a simplified model in which p takes only one of the three possible values 0, 1/2, and 1. The agents in group 0 (with $p=0$) make the opposite decision to the agents in group 1 (with $p=1$). We denote the group with $p=1/2$ as "group m." The probability of winning only depends on N_0, N_m, and N_1, which are the respective numbers of agents in group 0, m, and 1.

We begin by evaluating the average wealth reduction for the agents in each of the three groups. Let n be the number of agents in group m making the same decision (decision A) as those in group 0. N_m-n will then be the number of agents in group m making the same decision (decision B) as those in group 1. If $N_0+n<(N_m-n)+N_1$, or $n<N_m/2+(N_1-N_0)/2$, the agents making decision A will win; when $n>N_m/2+(N_1-N_0)/2$, the agents making decision B will win. The winner's wealth is increased by R, while the loser's wealth is reduced by 1. With N_0, N_m, and N_1 fixed, the probability of winning depends on n.

When $N_m \gg 1$, the distribution of n can be approximated by a Gaussian distribution

$$P(n) = \frac{1}{\sqrt{2\pi}\sigma_m} \exp[-(n-N_m/2)^2/(2\sigma_m^2)],$$

where $\sigma_m = \sqrt{N_m}/2$. Given the distribution, one can write down the average wealth change for the agents in group 0,

$$\Delta w_0 = R\int_0^{N_m/2+N_d/2} P(n)dn - \int_{N_m/2+N_d/2}^{N_m} P(n)dn,$$

where $N_d = N_1 - N_0$. This can be rewritten in term of the error function $\mathrm{erf}(x) = (2/\sqrt{\pi})\int_0^x e^{-t^2}dt$,

$$\Delta w_0 = -\frac{1-R}{2} + \frac{1+R}{2}\mathrm{erf}\left(\frac{N_d}{2\sqrt{2}\sigma_m}\right). \quad (1)$$

Similarly we can derive the average wealth change for the agents in group 1:

$$\Delta w_1 = -\frac{1-R}{2} - \frac{1+R}{2}\mathrm{erf}\left(\frac{N_d}{2\sqrt{2}\sigma_m}\right). \quad (2)$$

Since the numbers N_0 and N_1 are fluctuating, and on average N_0 and N_1 should be the same, we can average out the short time fluctuations of N_d. This is an *adiabatic approximation*. The average wealth change of the agents in groups 0 and 1 is given by $\Delta w_e = (N_0\Delta w_0 + N_1\Delta w_1)/(N_0+N_1)$. Substituting the expressions for Δw_0 and Δw_1, we have

$$\Delta w_e = -\frac{1-R}{2} - \frac{1+R}{2}\frac{N_d}{N_0+N_1}\mathrm{erf}\left(\frac{N_d}{2\sqrt{2}\sigma_m}\right). \quad (3)$$

Note that the second term in Δw_e, which is due to the fluctuations of N_d, is always negative [since $\mathrm{erf}(x)$ is an odd function]. When $N_0 \neq N_1$, the winning probabilities for making decisions A and B are not equal, and the market is not efficient (there is a short-time arbitrage opportunity). Thus this term can be interpreted as the cost due to market inefficiency. Large market inefficiency on average penalizes the players taking "extreme" positions.

For the agents in group m, if $n < N_m/2 + N_d/2$, then n agents in the group win, while $N_m - n$ agents in the group lose. On the other hand, if $n > N_m/2 + N_d/2$, then $N_m - n$ agents in the group win, but n agents lose. We need to take these two cases into account when evaluating the average:

$$\Delta w_m = \frac{1}{N_m}\left[\int_0^{N_m/2+N_d/2}[Rn-(N_m-n)]P(n)dn + \int_{N_m/2+N_d/2}^{N_m}[R(N_m-n)-n]P(n)dn\right].$$

After a few algebraic steps, we arrive at

$$\Delta w_m = -(1-R)/2 - \frac{1+R}{\sqrt{2\pi N_m}}\exp[-N_d^2/(2N_m)]. \quad (4)$$

The first term in Δw_m is the same as that in Δw_e. The second term can be interpreted as the market impact [7]. A large market impact (self-interaction) penalizes the cautious players; their own decisions increase their chances of being in the majority and hence increase their chances of losing.

To determine the transition from clustering to segregation, we need to calculate the distribution of N_d which will allow us to evaluate Δw_e and Δw_m. Let us denote the change in N_d in one time step as δN. On average $\delta N = 2N_0/(|d|/[(1-R)/2]) = N_0(1-R)/|d|$; this is the average number of extreme agents switching their strategies per time step (adaptation rate). The dynamics of N_d can be described as a random walk with mean reversal (there is a higher probability moving towards $N_d = 0$ than away from it). The individual step of the walk is given by $\pm\delta N$. The probability for changing from N_d to $N_d + \delta N$ is given by $W_+(N_d)$, and the probability for changing to $N_d - \delta N$ is given by W_-, where $W_\pm = \frac{1}{2}[1 \mp \mathrm{erf}(N_d/(2\sqrt{2}\sigma_m))]$. The steady state probability distribution $Q(N_d)$ for N_d should satisfy

$$Q(N_d) = W_-(N_d+\delta N)Q(N_d+\delta N) + W_+(N_d-\delta N)Q(N_d-\delta N). \quad (5)$$

For small δN one can convert the above equation to a differential equation. The solution of $Q(N_d)$ is given by

$$Q(N_d) \propto \exp\left[-\frac{2}{\delta N}\int_0^{N_d}\mathrm{erf}\left(\frac{n}{2\sqrt{2}\sigma_m}\right)dn\right]. \quad (6)$$

Now we average Δw_e and Δw_m over the distribution of $Q(N_d)$. We can easily obtain that

$$\Delta w_e = -\frac{1-R}{2} - \frac{(1+R)}{2}\frac{\delta N}{2(N_0+N_1)}. \quad (7)$$

Δw_m, on the other hand, is given by

$$\Delta w_m = -(1-R)/2 - \frac{1+R}{\sqrt{2\pi N_m}}\langle\exp[-N_d^2/(2N_m)]\rangle,$$

where the average is over the distribution $Q(N_d)$. This can be approximated as

$$\Delta w_m \sim -\frac{1-R}{2} - \frac{1+R}{\sqrt{2\pi}}\frac{1}{\sqrt{N_m+\sigma_d^2}},$$

since in the range $N_d < \sigma_m$, from which the main contribution to the average comes, $Q(N_d)$ can be well approximated by a Gaussian distribution centered at zero with the width $\sigma_d = \sqrt{\sqrt{2\pi}/2}\sqrt{\sigma_m\delta N}$. At the critical point, $N_0 = N_1 = N_m = N/3$, and $\Delta w_e = \Delta w_m$. It is easy to verify that this occurs when $\delta N \sim \sqrt{N_m}$. As $\delta N = N_0(1-R)/|d|$, the crossover value for $|d|$ is $|d_c| = A(1-R)\sqrt{N}$, where A is a constant of the order one.

The theory can be applied to a general EMG involving M groups of agents with $p = p_1, p_2, \ldots, p_M$ and a number of agents $N_1, N_2, \ldots, N_m$. The wealth reduction for a given Group l is given by (the derivation is given elsewhere)

$$\delta w_l = -\frac{(1-R)}{2} - \left(p_l - \frac{1}{2}\right)(1+R)\mathrm{erf}\left(\frac{N\left(\overline{p} - \frac{1}{2}\right)}{\sqrt{2}\sigma_m}\right) - \frac{\sqrt{2}(1+R)}{\sqrt{\pi}\sigma_m} p_l(1-p_l)\exp\left[-N^2\left(\overline{p} - \frac{1}{2}\right)^2/(2\sigma_m^2)\right],$$

where $\overline{p} = (1/N)\Sigma_j N_j p_j$ and $\sigma_m = \sqrt{\Sigma_j N_j p_j (1-p_j)}$. We can associate the second term with the market inefficiency (which affects mostly the extreme players with $p \sim 0$ or 1) and the third term with the market impact (which affects mostly the cautious players with $p = 1/2$). It is clear that $N(\overline{p} - \frac{1}{2})$ plays the role of N_d in the three-group EMG. The market inefficiency is measured by the fluctuation of $N(\overline{p} - \frac{1}{2})$. Consider the version of the original EMG in which the agent chooses a new p randomly when its wealth is below d. We can similarly argue that δN (the average change in $N\overline{p}$) is given by $\delta N \sim N(1-R)/|d|$. So we have $|d_c| = A(1-R)\sqrt{N}$, or equivalently,

$$N_c = \left[\frac{|d|}{A(1-R)}\right]^2.$$

This agrees very well with the numerical data. Figure 2 clearly shows that $N_c \propto d^2$, and Fig. 3 shows that $N_c/d^2 \propto 1/(1-R)^2$ when R is close to 1.

We can also understand the version of the model in which the new p value is chosen in the interval of width δp around the old p value. Since a smaller δp leads to a smaller δN, the cost due to market inefficiency is reduced. Thus smaller δp favors the "extreme" agents as $|d_c|$ is smaller; this is consistent with the results obtained in Ref. [11]. Reference [12] found that the periodic boundary condition used in the redistribution of the p value favors clustering. This is also not surprising. With the periodic boundary condition in p, δN is effectively increased, because some $p=0$ agents can switch to $p=1$ agents, even when δp is small.

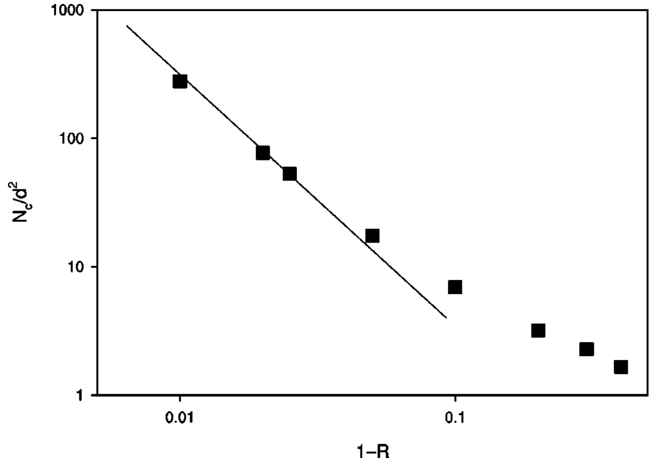

FIG. 3. N_c/d^2 vs $1-R$ for the original EMG with random redistribution.

In conclusion, we have derived a general formalism for studying the transition from clustering to segregation in the evolutionary minority game. The theory is based on an adiabatic approximation, in which the short-time fluctuations are integrated out to obtain a steady-state population distribution. We find that the effective rate of evolution plays an important role in determining the resulting steady-state population distribution. Frequent strategy switching leads to large market inefficiency that favors clustering of cautious agents. This result is quite universal, and it should provide much needed insight for studying the effect of evolution in agent-based models.

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